Indian Statistical Institute, Bangalore

B. Math (Hons.) Second Year

Second Semester - Ordinary Differential Equations

Final Exam — Date: 30th April 2024

Maximum marks: 50 — Duration: 3 hours

Section I: Answer any four and each question carries 6 marks.

1. Solve $xy'' = y' + (y')^3$.
2. Solve $y' = (1-x^2)^{\frac{-1}{2}}$ and use it to prove $\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2}\frac{1}{3\times 2^3} + +\frac{1\times 3}{2\times 4}\frac{1}{5\times 2^5} + \cdots$.
3. State and prove Sturm comparision Theorem.
4. Use Runge-Kutta fourth order method to find $y(0.2)$ upto three decimal places where $y' = x + y$ with $h = 0.1$.
5. Prove that $(0,0)$ is a asymptotically stable critical point of the system $x' = y$, $y' = -cy + d\sin x$ with $c, d > 0$.
6. Determine the stability of the critical points of $x' = 2y+2$ and $-y' = 5y+3x+2$.

Section II: Answer any two and each question carries 13 marks.

1. (a) Suppose y_1 and y_2 are twice continuously differentiable functions on $\mathbb{R}$ such that $y_1(0)y_2'(0) \neq y_2(0)y_1'(0)$. Is there an interval I containing 0 so that y_1 and y_2 are linearly independent solutions of a second order homogeneous linear differential equation on I. Justify your answer.

   (b) Solve $y'' - 3y' + 2y = 14\sin 2x - 18\cos 2x$ (*Marks: 7*).
2. (a) Find all Frobenius series solutions of $x^2y'' + xy' + (x^2-1)y = 0$ (*Marks: 6*).

   (b) Is the distance between any two consecutive positive zeros of solutions of $y'' + e^{-x^2}y' + (1+sin^2x)(1+cos^2x)y = 0$ bounded. Justify your answer.
3. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a sufficiently smooth function such that $xf(x) > 0$ for all $x \neq 0$ and $f(0) = 0$. Prove that $(0,0)$ is a stable critical point of the system $x' = y$ and $y' = -f(x)$ using Liapunov function.

   (b) Prove that the system $x' = 4x+4y-x(x^2+y^2)$ and $y' = -4x+4y-y(x^2+y^2)$ has a closed path C and all other paths approach C as $t \to \infty$ (*Marks: 7*).